# ROOTS OF UNITY FILTERING MUSICAL CHORDS

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Abstract. The mathematical theory of music has developed an extensive formal apparatus for the study of musical elements. Simplified approaches are required to address larger abstraction problems in current research topics such as discrete Fourier transform for scales and rhythms. Through the definition of group isomorphism, we translate from the triad chords over cyclic group $\mathbb{Z}_{12}$ to the roots of unity $(\zeta^{12}, \cdot)$, showing 8 equivalent algebraic and matrix expressions. Future work is needed to establish a mediated teaching-learning process that facilitates the assimilation and understanding of the structures underlying the chords.

**Keywords**: complex numbers, roots of unity, discrete Fourier transform, algebraic chord representations, music analysis, systematic musicology.

## 1. Introduction

The mathematical music theory has developed in recent years an vast corpus of formalization and research; as shown by the annual conference on mathematics and computation in music, which already has seven editions [MM19]. Given this advance, it is required that the research descend to the theoretical background and culture of the undergraduate musician student. Some efforts have been made in providing a wider musicological interest audience with the knowledge built up through books such as *Cold Math for Hot Music* [MMP16] and, more recently, *Theoretical And Practical Pedagogy Of Mathematical Music Theory* [MF18]. One of the recurring topics in the relationship between music and mathematics is complex numbers, the basis of complex analysis, which is used in countless physical and engineering applications. In music, this relationship is found in the use of Fourier transforms in the spectral analysis of signals. In this regard, an object of study developed by E. Amiot [Ami09, Ami16] focuses on the abstract sense of the discrete Fourier Transform (DFT). For this reason, the requirement to become familiar with terminology and analytical thinking is to deal with complex numbers and chords in a formalized way. The analogy between integers and complex numbers will be made first through the concept of group isomorphism from the triad chords [ASAALP$^+$12, Can18] and the roots of unity [Lan13]. Subsequently, the equivalent algebraic and vector representations of the chords will be exposed. As we will see, this paper provides an opportunity to advance the understanding of chord symmetries from a more formal perspective and the relationship between the integer world widely studied in group theory of music with complex numbers and DFT.

## 2. From integers to complex numbers: musical isomorphism

The usual association of chromatic scale notes with the group of integers module twelve $\mathbb{Z}_{12}$ favors calculation and notation. However, it would be interesting to take advantage of the more extensive properties of a closed field such as complex numbers

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$\mathbb{C}$. In that sense, it is proposed to transfer the labeling of the pitch class set theory through the isomorphism presented by the geometric representation of the unit's roots $\zeta^n$. Continuing the reasoning, there is an isomorphism between the group of integers module $n$ and the abelian cyclic group of the complex $n$ roots of the unit. Consequently, the association with the complex numbers to each particular note is given by means of the following definition of isomorphism between two abstract musical groups:

**Definition 2.1** (Musical Isomorphism)**.**

$$f : (\mathbb{Z}_{12}, +) \longrightarrow (\zeta^{12}, \cdot), n \mapsto i^n,$$

$$\text{It's a group homorphism } f(m+n) = i^{m+n} = i^m i^n = f(m)f(n)$$

$$\text{Then, if } f(m) = f(n),\ i^m = i^n \mapsto m \equiv n \pmod{12}.$$

Both groups are cyclic of order 12. On the other hand, this isomorphism can be corroborated by a Cayley table. From this isomorphism, two equivalent exponential definitions can be constructed: in the clockwise sense or in the opposite way. Although the Cartesian representation of the complex numbers is possible [Xen92], exponential manipulation is more useful, due to its analogy with the chord space in the circumference and as a preliminary study to the DFT [Ami16].

**Definition 2.2** (Form 1)**.** $C \mapsto e^{i\frac{\pi}{2}}$, $C\sharp \mapsto e^{i\frac{\pi}{3}}$, $D \mapsto e^{i\frac{\pi}{6}}$, $D\sharp \mapsto e^{i2\pi}$, $E \mapsto e^{i\frac{11\pi}{6}}$, $F \mapsto e^{i\frac{5\pi}{3}}$, $F\sharp \mapsto e^{i\frac{3\pi}{2}}$, $G \mapsto e^{i\frac{4\pi}{3}}$, $G\sharp \mapsto e^{i\frac{7\pi}{6}}$, $A \mapsto e^{i\pi}$, $A\sharp \mapsto e^{i\frac{5\pi}{6}}$, $B \mapsto e^{i\frac{2\pi}{3}}$.

**Definition 2.3** (Form 2)**.** $C \mapsto e^{-i\frac{3\pi}{2}}$, $C\sharp \mapsto e^{-i\frac{5\pi}{3}}$, $D \mapsto e^{-i\frac{11\pi}{6}}$, $D\sharp \mapsto e^{-i2\pi}$, $E \mapsto e^{-i\frac{\pi}{6}}$, $F \mapsto e^{-i\frac{\pi}{3}}$, $F\sharp \mapsto e^{-i\frac{\pi}{2}}$, $G \mapsto e^{-i\frac{2\pi}{3}}$, $G\sharp \mapsto e^{-i\frac{5\pi}{6}}$, $A \mapsto e^{-i\pi}$, $A\sharp \mapsto e^{-i\frac{7\pi}{6}}$, $B \mapsto e^{-i\frac{4\pi}{3}}$.

## 3. Triad chords in complex exponential form

Establishing the isomorphism between the group of integers and the twelfth roots of the unity and taking advantage of the intrinsic geometric analogy, we will build a structural equivalence from the definitions of triads on integers as seen in [ASAALP+12] and [Can18]. Therefore, the four chords are described in complex exponential form and their structural similarity in the group of integer modulo twelve. Capital letters $F$ are used for the major and augmented triad, and lowercase $f$ for the minor and diminished triad.

**Definition 3.1** (Four types of triads chords structure)**.**

$$\text{Major chords} = (F, F+4, F+7) \simeq (F, F \cdot e^{i\frac{4\pi}{3}}, F \cdot e^{i\frac{5\pi}{6}}) \simeq (F, F \cdot e^{-i\frac{2\pi}{3}}, F \cdot e^{-i\frac{7\pi}{6}}).$$

$$\text{Minor chords} = (f, f+3, f+7) \simeq (f, f \cdot e^{i\frac{3\pi}{2}}, f \cdot e^{i\frac{5\pi}{6}}) \simeq (f, f \cdot e^{-i\frac{\pi}{2}}, f \cdot e^{-i\frac{7\pi}{6}}).$$

$$\text{Augmented chords} = (F, F{+}4, F{+}8) \simeq (F, F{\cdot}e^{i\frac{4\pi}{3}}, F{\cdot}e^{i\frac{2\pi}{3}}) \simeq (F, F{\cdot}e^{-i\frac{2\pi}{3}}, F{\cdot}e^{-i\frac{4\pi}{3}}).$$

$$\text{Disminished} = (f, f+3, f+6) \simeq (f, f \cdot e^{i\frac{3\pi}{2}}, f \cdot e^{i\pi}) \simeq (f, f \cdot e^{-i\frac{\pi}{2}}, f \cdot f \cdot e^{-i\pi}).$$

3.1. **Exposure order of complex shapes.** The methodological procedure is to construct he finite group representations of the roots of unity associated with the major, minor, augmented and diminished triads. The exponential form of each chord mentioned is first represented and the derived algebraic forms are presented in the following order: Cartesian, polar form, trigonometric, vectorial, inverse exponential form, Cartesian matrix and trigonometric matrix representation. Finally the chord is shown in the conventional score.

**Example 3.2** (A major chord).

$$\zeta_{\mathbf{Amaj}} = \left[e^{i\pi}, e^{i\frac{\pi}{3}}, e^{i\frac{11\pi}{6}}\right] = \left[-1+0i, \frac{1}{2}+\frac{\sqrt{3}}{2}i, \frac{\sqrt{3}}{2}-\frac{1}{2}i\right] \tag{1}$$

$$= [1_{180^\circ}, 1_{60^\circ}, 1_{330^\circ}] = \left[\cos\pi + i\sin\pi, \cos\frac{\pi}{3}+i\sin\frac{\pi}{3}, \cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right] \tag{2}$$

$$= \left\{\begin{bmatrix}-1\\0\end{bmatrix}, \begin{bmatrix}\frac{1}{2}\\\frac{\sqrt{3}}{2}\end{bmatrix}, \begin{bmatrix}\frac{\sqrt{3}}{2}\\-\frac{1}{2}\end{bmatrix}\right\} = \left[e^{-i\pi}, e^{-i\frac{5\pi}{3}}, e^{-i\frac{\pi}{6}}\right] \tag{3}$$

$$= \left\{\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}, \begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}, \begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\-\frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}\right\} \tag{4}$$

$$= \left\{\begin{bmatrix}\cos\pi & -\sin\pi\\\sin\pi & \cos\pi\end{bmatrix}, \begin{bmatrix}\cos\frac{\pi}{3} & -\sin\frac{\pi}{3}\\\sin\frac{\pi}{3} & \cos\frac{\pi}{3}\end{bmatrix}, \begin{bmatrix}\cos\frac{11\pi}{6} & -\sin\frac{11\pi}{6}\\\sin\frac{11\pi}{6} & \cos\frac{11\pi}{6}\end{bmatrix}\right\} \tag{5}$$

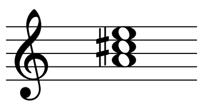

FIGURE 1. A major triad

**Example 3.3** (E minor chord).

$$\zeta_{\mathbf{Emin}} = \left[e^{i\frac{11\pi}{6}}, e^{i\frac{4\pi}{3}}, e^{i\frac{2\pi}{3}}\right] = \left[\frac{\sqrt{3}}{2}-\frac{1}{2}i, -\frac{1}{2}-\frac{\sqrt{3}}{2}i, -\frac{1}{2}+\frac{\sqrt{3}}{2}i\right] \tag{6}$$

$$= [1_{330^\circ}, 1_{240^\circ}, 1_{120^\circ}] = \left[\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}, \cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}, \cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right] \tag{7}$$

$$= \left\{\begin{bmatrix}\frac{\sqrt{3}}{2}\\\frac{-1}{2}\end{bmatrix}, \begin{bmatrix}\frac{-1}{2}\\\frac{-\sqrt{3}}{2}\end{bmatrix}, \begin{bmatrix}\frac{-1}{2}\\\frac{\sqrt{3}}{2}\end{bmatrix}\right\} = \left[e^{-i\frac{\pi}{6}}, e^{-i\frac{2\pi}{3}}, e^{-i\frac{4\pi}{3}}\right] \tag{8}$$

$$= \left\{\begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\\frac{-1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}, \begin{bmatrix}\frac{-1}{2} & \frac{\sqrt{3}}{2}\\\frac{-\sqrt{3}}{2} & \frac{-1}{2}\end{bmatrix}, \begin{bmatrix}-\frac{1}{2} & \frac{-\sqrt{3}}{2}\\\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}\right\} \tag{9}$$

$$= \left\{\begin{bmatrix}\cos\frac{11\pi}{6} & -\sin\frac{11\pi}{6}\\\sin\frac{11\pi}{6} & \cos\frac{11\pi}{6}\end{bmatrix}, \begin{bmatrix}\cos\frac{4\pi}{3} & -\sin\frac{4\pi}{3}\\\sin\frac{4\pi}{3} & \cos\frac{4\pi}{3}\end{bmatrix}, \begin{bmatrix}\cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3}\\\sin\frac{2\pi}{3} & \cos\frac{2\pi}{3}\end{bmatrix}\right\} \tag{10}$$

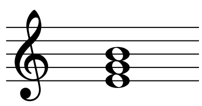

FIGURE 2. E minor triad

**Example 3.4** (A augmented chord)**.**

$$\zeta_{\mathbf{Aaug}} = \left[e^{i\pi}, e^{i\frac{\pi}{3}}, e^{i\frac{5\pi}{3}}\right] = \left[-1+0i, \frac{1}{2}+\frac{\sqrt{3}}{2}i, \frac{1}{2}-\frac{\sqrt{3}}{2}i\right] \tag{11}$$

$$= [1_{180^\circ}, 1_{60^\circ}, 1_{300^\circ}] = \left[\cos\pi + i\sin\pi, \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}, \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right] \tag{12}$$

$$= \left\{ \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix} \right\} = \left[e^{-i\pi}, e^{-i\frac{5\pi}{3}}, e^{-i\frac{7\pi}{6}}\right] \tag{13}$$

$$= \left\{ \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \right\} \tag{14}$$

$$= \left\{ \begin{bmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{bmatrix}, \begin{bmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix}, \begin{bmatrix} \cos\frac{5\pi}{3} & -\sin\frac{5\pi}{3} \\ \sin\frac{5\pi}{3} & \cos\frac{5\pi}{3} \end{bmatrix} \right\} \tag{15}$$

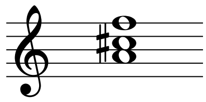

Figure 3. A augmented triad

**Example 3.5** (E diminished chord)**.**

$$\zeta_{\mathbf{Edis}} = \left[e^{i\frac{11\pi}{6}}, e^{i\frac{4\pi}{3}}, e^{i\frac{5\pi}{6}}\right] = \left[\frac{\sqrt{3}}{2} - \frac{1}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, -\frac{\sqrt{3}}{2} + \frac{1}{2}i\right] \tag{16}$$

$$= [1_{330^\circ}, 1_{240^\circ}, 1_{150^\circ}] = \left[\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}, \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}, \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right] \tag{17}$$

$$= \left\{ \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{-1}{2} \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} \\ \frac{-\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} \frac{-\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} \right\} = \left[e^{-i\frac{\pi}{6}}, e^{-i\frac{2\pi}{3}}, e^{-i\frac{4\pi}{3}}\right] \tag{18}$$

$$= \left\{ \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} \end{bmatrix}, \begin{bmatrix} \frac{-\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{-\sqrt{3}}{2} \end{bmatrix} \right\} \tag{19}$$

$$= \left\{ \begin{bmatrix} \cos\frac{11\pi}{6} & -\sin\frac{11\pi}{6} \\ \sin\frac{11\pi}{6} & \cos\frac{11\pi}{6} \end{bmatrix}, \begin{bmatrix} \cos\frac{4\pi}{3} & -\sin\frac{4\pi}{3} \\ \sin\frac{4\pi}{3} & \cos\frac{4\pi}{3} \end{bmatrix}, \begin{bmatrix} \cos\frac{5\pi}{6} & -\sin\frac{5\pi}{6} \\ \sin\frac{5\pi}{6} & \cos\frac{5\pi}{6} \end{bmatrix} \right\} \tag{20}$$

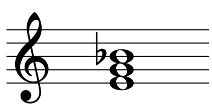

Figure 4. E diminished triad

## 4. Conclusions and future works

The purpose of these triad constructions were to pave the way for abstract DFT [Ami09, Ami16] in the musical sense of pitch classes to understand the current field of research in mathematical music theory. In this sense, this study constitutes the first approach to the subject.

The results show that multiple algebraic and matrix representations could make it difficult to grasp an elementary concept of musical theory such as chords. However, a multiplicity of inventive options unfold by applying the isomorphism between the group of integers and that of the roots of the unity of complex numbers. One of them, the description of other types of chords can be built from S. Cannas doctoral thesis [Can18], since is an immediate extension of this work and does not present major difficulties. We have to argue that the sense of familiarity with the complex expressions of the chords, facilitates the internalization of more abstract concepts from the beginning of musical training with a view to understanding the uses of discrete Fourier transform in Amiot's works.

The fundamental limitation lies in its pedagogical apparatus. Although for students more apt with mathematics, the transition between music and complex numbers can be smoother and immediately opens a world of creative questions and activities. Conversely, to the group least interested in mathematics, it may seem an excess of formalism or abstraction.

The need to graduate the transit of the chords of pitch classes to the group of the roots of the unity through a didactic process arises as a research topic in later studies. Future work should focus on building each complex group's algebraic, matrix, and vector representation on a planned proposal for a natural association between music and mathematics.

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